ANIKET ' S

OPERATION

MATHEMATICS

CBSE (XII) – 2023
LEVEL - II

(14th Revised Edition)

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PREFACE

*It is known to every student that **70 – 80 %** questions in **CBSE – XII** Exam have been asked from **NCERT** Text Book, and the remaining **20 – 30 %** are **HOTS** (**H**igh **O**rder **T**hinking **S**kills) questions.*

*After going through the market available books on Mathematics for **CBSE XII**, I come to know that all books contains a lot of **HOTS** questions which are **Out of Course** for **CBSE XII,** and students are wasting their valuable time on such questions.*

*As an outcome, in spite of Hard Labour students had performed below the level, what they expect from themselves, in their **Unit Tests** or **Formative Assessment Examinations**.*

*So, this '**Operation Mathematics CBSE XII – 2020 (Level – II)'** has been designed to provide some selective questions of **HOTS** for the target oriented preparation of **CBSE XII.***

*This package will not only bring confidence, but help the students in scoring the wonder tons **90 – 100 % Marks** in coming **CBSE XII – 2020 Exam.***

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1. Relation & Functions

Q. 1. Let N be the set of all natural numbers and R be the relation on $N \times N$ defined by **:** $(a, b)R(c, d) \Leftrightarrow ad(b + c) = bc(a + d)$ Show that relation R is an equivalence relation

Q. 2. If relation R is a defined on Q(set of rational numbers) by: $aRb \Leftrightarrow |a - b| \leq \frac{1}{2}$, then show that, R is not an equivalence relation.

Q. 3. Let $A = \{1, 2, - - - , 9\}$ & R be the relation in $A \times A$ defined by: $(a, b)R(c, d) \Leftrightarrow a + d = b + c$ Show that relation R is an equivalence relation. Also find the equivalence class $[(2, 5)]$.
$Ans: \{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9), \}$

Q. 4. Show that the relation R in the set $A = \{x: x\ is\ an\ integer\ \&\ 0 \leq x \leq 12\}$ as: $R = \{(a, b): a \equiv b(5)\}$, with $a \equiv b(5) \Rightarrow |a - b|$ is divisible by 5, is an equivalence relation. Also obtain the equivalence class $[0], [2]$. $Ans: [0] = \{0, 5, 10\}; [2] = \{2, 7, 12\}$

Q. 5. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and the relation R defined in A, by $R = \{(x, y): x, y\ either\ both\ odd\ or\ both\ even;\ x, y \in A\}$. Show that R is an equivalence relation. Write the equivalence classes of 1 & 2. $Ans: [1] = \{1, 3, 5, 7, 9\};\ [2] = \{2, 4, 6, 8\}$

Q. 6. Test whether relation R defined in $\boldsymbol{R}(real\ number)$ as $R = \{(a, b): a^2 - 4ab + 3b^2 = 0;\ a, b \in \boldsymbol{R}\}$ is reflexive, summative and transitive. $Ans: reflexive, not\ symmetric, not\ transitive$

Q. 7. If $f: [1\ \ \infty) \rightarrow [2\ \ \infty)$ is given by $(x) = x + \frac{1}{x}$, show that f is bijective.

Q. 8. Let $f: [-1, 1] \rightarrow R$ be a function defined by $f(x) = \frac{x}{x + 2}$. Prove that f is not bijective. Modify the co-domain only to make f bijective. $Ans: \left[-1, \frac{1}{3}\right]$

Q. 9. Let $f: [0, \infty) \rightarrow R$ be a function defined by $f(x) = 9x^2 + 6x - 5$. Prove that f is not bijective. Modify the co-domain only to make f bijective. $Ans: [-5\ \ \infty)$

Q. 10. Show that the function $f: R - \left\{\frac{7}{5}\right\} \rightarrow R - \left\{\frac{3}{5}\right\}$ defined by $f(x) = \frac{3x + 4}{5x - 7}$ is bijective.

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2. Inverse Trigonometrical Functions

Q. 1. Prove that, $tan^{-1}x = 2\,tan^{-1}[cosec(tan^{-1}x) - tan(cot^{-1}x)];\ x > 0$.

Q. 2. If, $cos^{-1}x + cos^{-1}y + cos^{-1}z = \pi$, Prove that, $x^2 + y^2 + z^2 + 2xyz = 1$.

Q. 3. If, $cos^{-1}\left(\frac{x}{a}\right) + cos^{-1}\left(\frac{y}{b}\right) = \alpha$. Prove that, $\frac{x^2}{a^2} - \frac{2xy}{ab}cos\alpha + \frac{y^2}{b^2} = sin^2\alpha$

Q. 4. If, $sin^{-1}x + sin^{-1}y + sin^{-1}z = \pi$, Prove that, $x\sqrt{1 - x^2} + y\sqrt{1 - y^2} + z\sqrt{1 - z^2} = 2xyz$

Q. 5. Solve for x, $(tan^{-1}x)^2 + (cot^{-1}x)^2 = \frac{5\pi^2}{8}$ $Ans: x = -1$

Q. 6. Prove that: $cos\,[tan^{-1}\{sin\,(cot^{-1}x)\}] = sin\,[cot^{-1}\{cos\,(tan^{-1}x)\}] = \frac{\sqrt{1 + x^2}}{\sqrt{2 + x^2}}$

Q. 7. Show that: $tan^{-1}\left\{\frac{\sqrt{1 + x^2} + \sqrt{1 - x^2}}{\sqrt{1 + x^2} - \sqrt{1 - x^2}}\right\} = \frac{\pi}{4} + \frac{cos^{-1}(x^2)}{2}$, $|x| < 1$

Q. 8. If, $y = cot^{-1}(\sqrt{\cos x}) - tan^{-1}(\sqrt{\cos x})$, prove that $sin\,y = tan^2\left(\frac{x}{2}\right)$.

Q. 9. Prove that: $tan\left\{\frac{\pi}{4} + \frac{1}{2}tan^{-1}\left(\frac{a}{b}\right)\right\} + tan\left\{\frac{\pi}{4} - \frac{1}{2}tan^{-1}\left(\frac{a}{b}\right)\right\} = \frac{2\sqrt{a^2 + b^2}}{b}$

Q. 10. Simplify: $tan^{-1}\left\{\frac{\sqrt{1 + sin\,x} + \sqrt{1 - sin\,x}}{\sqrt{1 + sin\,x} - \sqrt{1 - sin\,x}}\right\}$, $x \in [0\ \ \pi]$ $Ans: \frac{\pi}{2} - \frac{x}{2}, if x \in \left(0\ \ \frac{\pi}{2}\right)\ \&\ x\ if x \in \left(\frac{\pi}{2}\ \ \pi\right)$

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3. Matrices

Q. 1. If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ and $f(x) = x^2 - x + 2$, then show that $f(A) = O$. Hence find A^3 and A^{-1}.

$Ans: A^3 = \begin{bmatrix} -5 & 2 \\ -4 & 0 \end{bmatrix}, A^{-1} = \frac{1}{2}\begin{bmatrix} -2 & 2 \\ -4 & 3 \end{bmatrix}$

Q. 2. Find the value of x, y, z if the matrix $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfies the condition $A^{-1} = A^T$

$Ans: x = \pm\frac{1}{\sqrt{2}}, y = \pm\frac{1}{\sqrt{6}}, z = \pm\frac{1}{\sqrt{3}}$

Q. 3. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$. Hence find the value of A^5. $Ans: \begin{bmatrix} 62 & 149 \\ -149 & 87 \end{bmatrix}$

Q. 4. Find the matrix A so that $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $Ans: \begin{bmatrix} -3 & -2 \\ 19 & 12 \end{bmatrix}$

Q. 5. If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$, find the value of k so that $A^2 = kA - 3I$ $Ans: No\ real\ k\ exists$

Q. 6. Find the matrix X such that,

$(i)\ X\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$ $(ii)\ \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$ $Ans: (i) \begin{bmatrix} -8 & 4 \\ -19 & 12 \end{bmatrix}\ (ii) \begin{bmatrix} -3 & -14 \\ 4 & 17 \end{bmatrix}$

Q. 7. Using Principle of Mathematical Induction, prove that $A^n = \begin{bmatrix} a^n & \frac{b(a^n - 1)}{a - 1} \\ 0 & 1 \end{bmatrix}$ if, $A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$

Q. 8. If $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, find value(s) of x, y such that $(xI + yA)^2 = A$

$Ans: x = y = \pm\frac{1}{\sqrt{2}}$ $x = \pm\frac{1}{\sqrt{2}}, y = \mp\frac{1}{\sqrt{2}}$

Q. 9. Find the matrix X so that $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} X = \begin{bmatrix} -7 & 2 \\ -8 & 4 \\ -9 & 6 \end{bmatrix}$ $Ans: \begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix}$

Q. 10. If $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix}$, then prove that, $A^3 = pI + qA + rA^2$

Q. 11. If, $A = \begin{bmatrix} 2 & 4 & -1 \\ -1 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}$. Show that $(AB)^T = B^T . A^T$

Q. 12. If, $A = \begin{bmatrix} 2 & 4 & -1 \\ -1 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}$. Find $B^{-1}.A^{-1}$ $Ans: \frac{1}{15}\begin{bmatrix} 2 & 15 \\ 1 & 0 \end{bmatrix}$

Q. 13. If matrix, $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, satisfies the quadratic, $x^2 + ax + b = 0$, find the numbers a and b

Hence find the value of $A^4 - 4A^3 - A^2 + 2A - 5I$. $Ans: \begin{bmatrix} -44 & -20 \\ 20 & -24 \end{bmatrix}$

Q. 14. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ satisfies $(A + B)^2 = A^2 + B^2$. Find the value of 'a' and 'b'.

$Ans: a = 1, b = 4$

Q. 15. For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the numbers '$a$' and '$b$' if $A^2 + aA + bI = 0$, hence find A^{-1}.

$Ans: a = -4, b = 1; A^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

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4. Determinants

Q. 1. If $D = \begin{vmatrix} 1 & a & a^2 \\ a & a^2 & 1 \\ a^2 & 1 & a \end{vmatrix} = -4$, find the value of $\begin{vmatrix} a^3 - 1 & 0 & a - a^4 \\ 0 & a - a^4 & a^3 - 1 \\ a - a^4 & a^3 - 1 & 0 \end{vmatrix}$ $Ans: 16$

Q. 2. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ & $B = \begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix}$, then prove that $adj(AB) = adj(B).adj(A)$

Q. 3. If, $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$ & $G(y) = \begin{bmatrix} \cos y & 0 & -\sin y \\ 0 & 1 & 0 \\ \sin y & 0 & \cos y \end{bmatrix}$ then prove that

$(i)[F(x)]^{-1} = F(-x)$ $\quad$ $(ii)[G(y)]^{-1} = G(-y)$ $\quad$ $(iii)[F(x).G(y)]^{-1} = G(-y).F(-x)$.

Q. 4. If, $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ then prove that $A.adj(A) = adj(A).A = |A|I$.

Q. 5. Let, $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$. Show that $AB = BA = 4I$, hence solve the system of equations $x + y + 2z = 0, 3x + 2y + z = 7, 2x + y + 3z = 2$ $\quad$ $Ans: x = \frac{13}{4}, y = -\frac{3}{4}, z = -\frac{5}{4}$.

Q. 6. If, $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$. Find A^{-1}. Using A^{-1} solve the system of linear equations:

$x + 2y + z = 4, -x + y + z = 0, x - 3y + z = 2.$ $\quad$ $Ans: x = \frac{18}{10}, y = \frac{4}{10}, z = \frac{14}{10}$

Q. 7. If, A = $\begin{bmatrix} -3 & 5 & 2 \\ 2 & -4 & 3 \\ 1 & -2 & 1 \end{bmatrix}$. Find A^{-1}. Using A^{-1} solve the system of linear equations:

$2x - 3y + 5z = 11, 3x + 2y - 4z = -5, x + y - 2z = -3.$ $\quad$ $Ans: x = 1, y = 2, z = 3$

Q. 8. If, A = $\begin{bmatrix} 2 & 1 & -1 \\ 4 & -3 & 2 \\ 3 & -2 & 3 \end{bmatrix}$. Find A^{-1}. Using A^{-1} solve the system of linear equations:

$3x - 2y + 3z = 8, 2x + y - z = 1$ and $4x - 3y + 2z = 4,$ $\quad$ $Ans: x = 1, y = 2, z = 3$

Q. 9. A total amount of $Rs. 7{,}000$ is deposited in three different savings bank accounts with annual interest rates of $5\%, 8\%$ and $8\frac{1}{2}\%$ respectively. The total annual interest from these three accounts is $Rs. 550$. Equal amounts have been deposited in the 5% and 8% savings accounts. Find the amount deposited in each of the three accounts, with the help of matrices. $\quad$ $Ans: 1125, 1125, 4750$

Q. 10. A school wants to award its students for the values of Honesty, Regularity and Hard work with a total cash award of $Rs. 6000$. Three times the award money for Hard work added to that given for Honesty amounts to $Rs\ 11000$. The award money given for Honesty and Hard work together is double the one given for Regularity. Represent the above situation algebraically and find the award money for value, using matrix method.

$Ans: Honesty = Rs. 1000,\ Regularity = Rs. 2000, Hard\ work = Rs. 3000$

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5. Differentiations

Q. 1. Find value of a if the function $f(x) = \begin{cases} a \sin\left\{\frac{\pi(x+1)}{2}\right\} & ; x \leq 0 \\ \frac{\tan x - \sin x}{x^3} & ; x > 0 \end{cases}$ is continuous at $x = 0$

$Ans: a = \frac{1}{2}$

Q. 2. Test the continuity of the function $f(x) = \begin{cases} (x-a)\sin\left(\frac{1}{x-a}\right) & ; x \neq a \\ 0 & ; x = a \end{cases}$

$Ans: Always\ Continuous$

Q. 3. Find value of a, b if for the function $f(x) = \begin{cases} 5ax - 2b & ; x < 1 \\ 11 & ; x = 1 \\ 3ax + b & ; x > 1 \end{cases}$ is continuous at $x = 1$

$Ans: a = 3, b = 2$

Q. 4. Find values of a, b if the function $f(x) = \begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x} & ; x < \dfrac{\pi}{2} \\ a & ; x = \dfrac{\pi}{2} \\ \dfrac{b(1-\sin x)}{(\pi-2x)^2} & ; x > \dfrac{\pi}{2} \end{cases}$ is continuous at $x = \dfrac{\pi}{2}$

$Ans: a = \frac{1}{2}, b = 4$

Q. 5. Find value of k if the function $f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2} & ; x < 0 \\ k & ; x = 0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} & ; x > 0 \end{cases}$ is continuous at $x = 0$ $Ans: k = 8$

Q. 6. Find value of a, b if the function $f(x) = \begin{cases} \dfrac{x-4}{|x-4|} + a & ; x < 4 \\ a + b & ; x = 4 \\ \dfrac{|x-4|}{x-4} + b & ; x > 4 \end{cases}$ is continuous at $x = 4$

$Ans: a = 1, b = -1$

Q. 7. Find value of a, b, c if the function $f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x} & ; x < 0 \\ c & ; x = 0 \\ \dfrac{\sqrt{x+bx^2} - \sqrt{x}}{b\, x^{3/2}} & ; x > 0 \end{cases}$ is continuous at $x = 0$

$Ans: a = -\frac{3}{2}, c = \frac{1}{2}, b = real\ number\ other\ than\ 0$

Q. 8. Find the values of 'k' so that the function, $f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1}; & x \neq 0 \\ k & ; x = 0 \end{cases}$ is continuous at $x = 0$

$Ans: No\ real\ value\ of\ k\ can\ make\ the\ function\ continuous\ at\ x = 0$

Q. 9. Let $f(x+y) = f(x) + f(y)\, \forall x, y \in R$. If $f(x)$ is continuous at $x = 0$. Then show that $f(x)$ is continuous at all real x.

Q. 10. Prove that the function defined by $f(x) = |x-a|$; $x \in Real$, is not differentiable at $x = a$.

Q. 11. If $x^y = e^{x-y}$, then prove that $\dfrac{dy}{dx} = \dfrac{\log x}{(\log ex)^2}$

Q. 12. If $y = \tan^{-1}\left\{\dfrac{\sqrt{1+\sqrt{x}}+\sqrt{1-\sqrt{x}}}{\sqrt{1+\sqrt{x}}-\sqrt{1-\sqrt{x}}}\right\}$, then prove that $\dfrac{dy}{dx} = \dfrac{-1}{4\sqrt{x-x^2}}$.

Q. 13. If $x = \sin t$, $y = \sin at$, prove that $(1-x^2)y_2 - xy_1 + a^2 y = 0$

Q. 14. If $y = \sin^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{2}\right)$ then prove that $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{1-x^2}}$

Q. 15. If $y = \sin^{-1}\left(\dfrac{2^{x+1}.3^x}{1+(36)^x}\right)$; then prove that $\dfrac{dy}{dx} = \dfrac{2^{x+1}.3^x.\log 6}{1+(36)^x}$

Q. 16. If $x^2 + y^2 = t - \dfrac{1}{t}$ and $x^4 + y^4 = t^2 + \dfrac{1}{t^2}$ prove that $\dfrac{dy}{dx} = \dfrac{1}{x^3 y}$.

Q. 17. If $x^m y^n = (x+y)^{m+n}$ prove that $(i)\ \dfrac{dy}{dx} = \dfrac{y}{x}$ $\quad (ii)\ \dfrac{d^2y}{dx^2} = 0$

Q. 18. If $y = \sin^{-1}\{x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\}$ prove that $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}} - \dfrac{1}{2\sqrt{x-x^2}}$

Q. 19. If $y = \left(x + \sqrt{1+x^2}\right)^m$, then prove that $(1+x^2)y_2 + x\,y_1 - m^2 y = 0$

Q. 20. If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$ then prove that, $\dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

Q. 21. If $f(x) = cot^{-1}\left(\frac{x^x - x^{-x}}{2}\right)$ prove that $f^{/}(1) = -1$.

Q. 22. If $y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c}{(x-c)} + 1$,

prove that, $\frac{dy}{dx} = \frac{y}{x}\left\{\frac{a}{(a-x)} + \frac{b}{(b-x)} + \frac{c}{(c-x)}\right\}$

Q. 23. If $y = f\left(\frac{2x-1}{x^2+1}\right)$ and $\frac{df(t)}{dt} = sin^2 t$, then prove that $\frac{dy}{dx} = \left\{\frac{2(1+x-x^2)}{(x^2+1)^2}\right\} sin^2\left(\frac{2x-1}{x^2+1}\right)$

Q. 24. If $y = x\,log\left(\frac{x}{a+bx}\right)$, then prove that, $x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$

Q. 25. Prove that $\frac{d}{dx}\left\{\frac{1}{4\sqrt{2}}log\left(\frac{x^2 + x\sqrt{2}+1}{x^2 - x\sqrt{2}+1}\right) + \frac{1}{2\sqrt{2}}tan^{-1}\left(\frac{x\sqrt{2}}{1-x^2}\right)\right\} = \frac{1}{1+x^4}$

Q. 26. If $y = e^x\,tan^{-1}x$, then prove that $(1+x^2)y_2 - 2(1-x+x^2)y_1 + (1-x)^2 y = 0$

Q. 27. Differentiate $tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ with respect to $sin^{-1}\left(2x\sqrt{1-x^2}\right)$ if $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ $\quad Ans: \frac{1}{2}$

Q. 28. Differentiate $tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $sin^{-1}\left(\frac{2x}{1+x^2}\right)$ if $-1 < x < 1$ $\quad Ans: \frac{1}{4}$

Q. 29. If $x = sec\,t - cos\,t$ and $y = sec^n t - cos^n t$, prove that $\left(\frac{dy}{dx}\right)^2 = n^2\left(\frac{4+y^2}{4+x^2}\right)$

Q. 30. If $y = log\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$ then prove that $x(x+1)^2 y_2 + (x+1)^2 y_1 = 2$

Q. 31. If $y = x^{x^x}$, prove that $xyy_2 - xy_1{}^2 - y^2 = 0$

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6. Applications of Derivatives

Q. 1. A point source of light along a straight road is at a height of a meteres. A boy b meters in height is walking along the road, away from light at rate of c $meter/min$. Find the rate at which (i) the shadow is lengthening. (ii) The tip of the shadow moves. $\quad Ans: (i)\ \frac{bc}{a-b}\ m/\text{min}\ (ii)\ \frac{ac}{a-b}\ m/\text{min}$

Q. 2. Water is leaking from a conical funnel at the rate $5\ cm^3/s$. If the radius of the base of the funnel is $5\ cm$ and its altitude is $10\ cm$, find the rate at which water level is dropping when it is $2.5\ cm$ from the top. $\quad Ans: \frac{16}{45\pi}\ m/min$

Q. 3. A man is moving away from a tower $41.6m$ high at the rate of $2m/sec$. Find the rate at which the angle of elevation of the top of the tower is changing when he is at a distance of $30m$ from the foot of the tower. Assume that the eye level of the man is $1.6m$ from the ground. $\quad Ans: -0.032^c/s$

Q. 4 A kite is $120\ m$ high and $130\ m$ of string is out. If the kite is moving away horizontally at the rate of $52\ m/s$. Find the rate at which the string is being paid out. Assuming there is no slack in the string. $\quad Ans: 20\ m/s$

Q. 5. A balloon in the form of a right circular cone surmounted by a hemisphere, having a diameter equal to the height of the cone, is being inflated by pumping in gas. How fast is its volume changing with respect to its total height, when the height of the balloon is $9cm$. $\quad Ans: 12\pi\ cm^3/cm$

Q. 6. Find the interval in which the function $f(x) = \sin^4 x + \cos^4 x$, $x \in \left[0\ \ \frac{\pi}{2}\right]$ is

(i) Increasing (ii) Decreasing. $\quad Ans: (i)\left[\frac{\pi}{4}\ \ \frac{\pi}{2}\right] (ii)\left[0\ \ \frac{\pi}{4}\right]$

Q. 7. Find the interval in which the function $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$ is

(i) strictly increasing (ii) strictly decreasing. $\quad Ans: (i)(1,\ 2) \cup (3,\ \infty)\ \ (ii)(-\infty,\ 1) \cup (2,\ 3)$

Q. 8. Prove that for $0 < k < \frac{1}{3}$, the function $f(x) = kx^3 - 9kx^2 + 9x + 3$ is increasing.

Q. 9. Prove that for $-\infty < k < -3$, the function $f(x) = (k+2)x^3 - 3kx^2 + 9kx - 1$, is decreasing.

Q. 10. Find the interval in which the function $f(x) = log(cos\, x)$, is (i) strictly increasing and (ii) strictly decreasing $\quad Ans: (i) \left(-\frac{\pi}{2},\ 0\right)\ (ii) \left(0,\ \frac{\pi}{2}\right)$

Q. 11. The sum of length of hypotenuse and a side of a triangle is given. Show that the area of the triangle is maximum when the angle between them is $\frac{\pi}{3}$

Q. 12. A rectangle is inscribed in a semi – circle of radius R with one of its side on the diameter of the semi – circle. Find the dimension of the rectangle so that the rectangle has maximum area.

$Ans: \sqrt{2}R, \frac{R}{\sqrt{2}}$

Q. 13. A wire of length $36m$ is to be cut into two pieces. One of the pieces is to be made into a square and other into an equilateral triangle. What should be the length of the each piece so that the combined area of the square and the equilateral triangle is minimum? $\quad Ans: \frac{144\sqrt{3}}{9+4\sqrt{3}}, \frac{324}{9+4\sqrt{3}}$

Q. 14. An open box with square base is to be made out of a given quantity of cardboard of area c^2sq. units Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ $cubic\ units$.

Q. 15. A given quantity of metal is to be cast into a half cylinder (a rectangular base and semi – circular ends). Show that the total surface area will be least when the ratio of the length of the cylinder to the diameter of the ends is $\pi : (\pi + 2)$

Q. 16. The length of sides of an isosceles triangle are $9 + x^2$, $9 + x^2$ and $18 - 2x^2$ units. Find the value of x, which makes the area of triangle maximum. $\quad Ans: x = \sqrt{3}\ units$

Q. 17. Find the area of the greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$Ans: 2ab\ sq.\ units$

Q. 18. Find the equation of the line through the point $(3, 4)$ which cuts off from the first quadrant a triangle of minimum area. $\quad Ans: 4x + 3y = 24$

Q. 19. Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible, when revolved about one of its sides. Also find the maximum volume.

$Ans: 12cm \times 6cm, 72\ cm^2$

Q. 20. The sum of surface area of a cube and a sphere is given. Show that when the sum of their volume is least, the diameter of the sphere is equal to the edge of the cube.

Q. 21. Find the minimum distance between the curve $x^2 = 8y$ and point $(2,\ 4)$. $\quad Ans: 2\sqrt{2}$

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7. Indefinite Integrations

Q. 1. Prove that : $\int \sqrt{\frac{x-1}{2-x}}\, dx = -\sqrt{3x - x^2 - 2} + \frac{1}{2} sin^{-1}(2x - 3) + C$

Q. 2. Prove that: $\int x^2 . sin^{-1}(x) dx = \frac{x^3 sin^{-1} x}{3} + \frac{\sqrt{1-x^2}.(2+x^2)}{9} + C$

Q. 3. Prove that : $\int \frac{dx}{1 + 3e^x + 2e^{2x}} = log\left|\frac{e^x(e^x+1)}{(2e^x+1)^2}\right| + C$

Q. 4. Prove that : $\int \frac{dx}{sin\, x - sin\, 2x} = -\frac{1}{2} log|1 - \cos x| - \frac{1}{6} log|1 + \cos x| + \frac{2}{3} log|1 - 2\cos x| + C$

Q. 5. Prove that : $\int \sqrt{tan\, x}\, dx = \frac{1}{\sqrt{2}} tan^{-1}\left(\frac{tan\, x - 1}{\sqrt{2 tan\, x}}\right) + \frac{1}{2\sqrt{2}} log\left(\frac{tan\, x - \sqrt{2 tan\, x} + 1}{tan\, x + \sqrt{2 tan\, x} + 1}\right) + C$

Q. 6. Prove that : $\int \frac{dx}{\sqrt{1 + 3e^x + e^{2x}}} = x - log\left|2 + 3e^x + 2\sqrt{1 + 3e^x + e^{2x}}\right| + C$

Q. 7. Prove that: $\int \sqrt{\frac{\sin\,(x-a)}{\sin\,(x+a)}}\,dx$

$$= -\cos a\,.\sin^{-1}(\sec a.\cos x) - \sin a.\log\left|\sin x + \sqrt{\sin^2 x - \sin^2 a}\right| + C$$

Q. 8. Prove that : $\int \frac{dx}{\sin x + \sec x} = \frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\sin x - \cos x}{\sqrt{3}-\sin x+\cos x}\right| + \tan^{-1}(\sin x + \cos x) + C$

Q. 9. Prove that : $\int \sin^{-1}\sqrt{\frac{x}{x+a}}\,dx = (x+a).\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + C$

Q. 10. Prove that : $\int \sec^3 x\,dx = \frac{1}{2}\{\sec x.\tan x + \log|\sec x + \tan x|\} + C$

Q. 11. Prove that : $\int \frac{x^2}{(x\sin x + \cos x)^2}\,dx = \frac{\sin x - x\cos x}{x\sin x + \cos x} + C$

Q. 12. Prove that : $\int \frac{dx}{1+\cot^3 x} = \log\left|\frac{(\sin x + \cos x)^3}{(2+\sin 2x)^2}\right| + x + C$

Q. 13. Prove that : $\int \frac{dx}{1+x^4} = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}\,x}\right) - \frac{1}{4\sqrt{2}}\log\left(\frac{x^2-\sqrt{2}\,x+1}{x^2+\sqrt{2}\,x+1}\right) + C$

Q. 14. Prove that : $\int \frac{\sqrt{\cos 2x}}{\sin x}\,dx = \sqrt{2}\log\left|\sqrt{2}\cos x - \sqrt{\cos 2x}\right| - \log\left|\cot x + \sqrt{\cot^2 x - 1}\right| + C$

Q. 15. Prove that : $\int \left(\frac{\sin 4x - 2}{1-\cos 4x}\right)e^{2x}\,dx = \frac{1}{2}e^{2x}\cot 2x + C$

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8. Definite Integrations

Prove the following integrals

Q. 1. $\int_0^1 \left(\frac{\log x}{\sqrt{1-x^2}}\right)dx = -\frac{\pi}{2}\log 2$

Q. 2. $\int_0^a \frac{dx}{x+\sqrt{a^2-x^2}} = \frac{\pi}{4}$

Q. 3. $\int_0^1 \cot^{-1}(1-x+x^2)\,dx = \frac{\pi}{2} - \log 2$

Q. 4. $\int_0^1 \left\{\frac{\log(1+x)}{1+x^2}\right\}dx = \frac{\pi}{8}\log 2$

Q. 5. $\int_0^{\pi/2} \left(\frac{x}{\sin x + \cos x}\right)dx = \frac{\pi}{2\sqrt{2}}\log\left(\sqrt{2}+1\right)$

Q. 6. $\int_0^{2\pi} \frac{dx}{1+e^{\cos x}} = \pi$

Q. 7. $\int_0^{\infty} \frac{x\,dx}{(1+x)(1+x^2)} = \frac{\pi}{4}$

Q. 8. $\int_{-1/2}^{1} |x\cos(\pi x)|\,dx = \frac{3}{2\pi} - \frac{1}{\pi^2}$

Q. 9. $\int_{-\pi/2}^{\pi/2} \left\{\frac{\cos x}{1+e^x}\right\}dx = 1$

Q. 10. $\int_0^{\pi} \frac{dx}{a^2 - 2a\cos x + 1} = \frac{\pi}{a^2-1}$

Q. 11. $\int_0^{\pi} \frac{x\,dx}{1-\cos a\sin x} = \frac{\pi(\pi-a)}{\sin a}$

Q. 12. $\int_0^{\pi/2} \frac{dx}{1+\sin x\cos x} = \frac{2\pi}{3\sqrt{3}}$

Q. 13. $\int_{-1}^{1} \frac{x^3+|x|+1}{x^2+|x|+1}\,dx = \log 3 + \frac{\pi}{3\sqrt{3}}$

Q. 14. $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\,dx = \pi^2$

Q. 15. $\int_0^1 x.(\tan^{-1}x)^2\,dx = \frac{\pi}{16}(\pi-4) + \log\sqrt{2}$

Q. 16. $\int_0^{\pi/2} \left(\frac{\sin^2 x}{\sin x + \cos x}\right)dx = \frac{1}{\sqrt{2}}\log\left(\sqrt{2}+1\right)$

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9. Area Bounding

Using the Method of Integration

Q. 1. Find the area of the region enclosed between the curves $x + y^2 - 1 = 0$ & $x - y^2 + 1 = 0$. Ans: $\frac{8}{3}$

Q. 2. Find the area bounded by the curves $y = 6x - x^2$ and $y = x^2 - 2x$ Ans: $\frac{64}{3}$ sq. units.

Q. 3. Find the area between the curves, $(y-2)^2 = x-1,\ 2y - x = 4$ & $y = 0$ $Ans: 9\ sq\ units.$

Q. 4. Find the area of the region bounded by the curves $x + y \le 6, x^2 + y^2 \le 6y$ & $y^2 \le 8x$.

Ans: $\frac{1}{12}(27\pi - 2)$ sq. units.

Q. 5. Find the area of region $\{(x, y): 0 \le y \le x^2 + 3, 0 \le y \le 2x + 3, 0 \le x \le 3\}$. Ans: $\frac{50}{3}$

Q. 6. Find the area bounded between the curves: $y^2 \ge ax;\ x^2 + y^2 \le 2ay; x, y \ge 0$ $Ans: \left\{\frac{\pi}{4} - \frac{2}{3}\right\} a^2$

Q. 7. Find the area enclosed between the curves $y = |x-1|$ and $y = 3 - |x|$ Ans: 4 sq. units.

Q. 8. Find the area of the region between the curves $y^2 = x$ and $x + y = 2$ Ans: $\frac{9}{2}$ sq. units

Q. 9. Using method of integration find the area of the triangle ABC, co-ordinates of whose vertices are $A(1, -2), B(3, 5)$ and $C(5, 2)$. $Ans: 10\ sq.units$

Q. 10. Find the area bounded between curves $y = \sqrt{5 - x^2}$ & $y = |x - 1|$ $Ans: \left(\frac{5\pi}{4} - \frac{1}{2}\right) sq.units$

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10. Differential Equation

Find the general or particular solution of the following differential equations:

Q. 1. $\frac{dy}{dx} = (1 + x + y + xy)$ $Ans: log(1 + y) = x + \frac{x^2}{2} + C$

Q. 2. $\frac{dy}{dx} = (4x + y + 1)^2$ $Ans: tan^{-1}\left(\frac{4x + y + 1}{2}\right) = 2x + C$

Q. 3. $\frac{dy}{dx} = sin(x + y) + cos(x + y)$ $Ans: 1 + tan\left(\frac{x + y}{2}\right) = Ce^x$

Q. 4. $sin^{-1}\left(\frac{dy}{dx}\right) = x + y$ $Ans: x = tan(x + y) - sec(x + y) + C$

Q. 5. $(2x + 2y + 3)dy = (x + y + 1)dx$ $Ans: 6x + 6y + log|3x + 3y + 4| = 9x + C$

Q. 6. $\frac{dy}{dx} = \frac{y^2 - 2xy - x^2}{y^2 + 2xy - x^2}; y(1) = -1$ $Ans: x + y = 0$

Q. 7. $y^2 cos\left(\frac{y}{x}\right) dx = x\left\{x\, sin\left(\frac{y}{x}\right) + y\, cos\left(\frac{y}{x}\right)\right\} dy$ $Ans:\ y\, sin\left(\frac{y}{x}\right) = C$

Q. 8. $x^2 y\, dx + (x^3 + y^3)dy$ $Ans: log\, y - \frac{1}{3}\left(\frac{x}{y}\right)^3 = C$

Q. 9. $x\, cos\left(\frac{y}{x}\right).(ydx + xdy) = ysin\left(\frac{y}{x}\right).(xdy - ydx)$ $Ans: sec\left(\frac{y}{x}\right) = Cxy$

Q. 10. $(x^2 - xy + y^2)dy + y^2 dx = 0\, ; y(1) = 1$ $Ans: tan^{-1}\left(\frac{x}{y}\right) = \frac{\pi}{4} - \log y$

Q. 11. Show that the differential equation $x^2 \frac{dy}{dx} - xy = 1 + cos\left(\frac{y}{x}\right)$ is non – homogenous, then solve it.

$Ans: tan\left(\frac{y}{2x}\right) = C - \frac{1}{2x^2}$

Q. 12. $(sin^{-1}y - x)dy = \sqrt{1 - y^2}dx; y(0) = 0$ $Ans:\ x = sin^{-1}y - 1 + e^{-sin^{-1}y}$

Q. 13. $(1 + xy)ydx + (1 - xy)xdy = 0,\ y(1) = 1$ $Ans: ex = ye^{\frac{1}{xy}}$

Q. 14. $xdy = (x^3 y^6 - y)dx$ $Ans: \frac{2}{(xy)^5} = \frac{5}{x^2} + C$

Q. 15. $\sqrt{1 + x^2 + y^2 + x^2y^2} + xy\frac{dy}{dx} = 0.$

$Ans: \sqrt{1 + x^2} + \sqrt{1 + y^2} + log\left|\sqrt{1 + x^2} - 1\right| - log|x| + C$

Q. 16. The temperature T of an object decreases at a rate which is proportional to the difference $T-S$, where S is the constant temperature of the surrounding medium. Solve the differential equation if it is given that the temperature of surrounding is 50^oC, and the object cools from 100^oC to 80^oC in 4 seconds. Find the temperature of the object after 8 seconds. $Ans: 68^{\circ}$

Q. 17. The rate of growth of a population is proportional to the number present. If the population of a city doubled in the past 25 years, and the present population is 1,00,000, when will the city have a population of 5,00,000? $(log\,5 = 1.609, log\,2 = 0.6931)$ $Ans: 58\ years\ from\ now.$

Q. 18. Find the differential equation of all circles of radius r. $Ans: \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2$

Q. 19. Find the differential equation of the curve $y = ae^{mx} + be^{nx}$, with a & b the parameters.
$Ans: y_2 - (m+n)y_1 + mny = 0$

Q. 20. Find the differential equation of the curve $y = a\,cos(log\,x) + b\,sin(log\,x)$, with a & b the parameters. $Ans: x^2y_2 + xy_1 + y = 0$

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11. Vectors

Q. 1. Decompose the vector $5i-2j+5k$ into vectors which are parallel & perpendicular to $3i + k$.
$Ans: 6i + 2k, -i-2j+3k$

Q. 2. If a vector $\vec{a}$ of magnitude 4 $units$ makes angles $\frac{\pi}{4}$ with i, $\frac{\pi}{3}$ with j and an obtuse angle θ with k, then find θ and hence the vector $\vec{a}$. $Ans: \frac{2\pi}{3}, 2(\sqrt{2}i + j - k)$

Q. 3. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors such that $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c} = 0$ and angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{6}$, then prove that $\vec{a} = \pm 2(\vec{b}\times\vec{c})$

Q. 4. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors such that $\vec{b}\times\vec{c} = \vec{a}$ and $\vec{a}\times\vec{b} = \vec{c}$. Prove that $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular to each other and $|\vec{b}| = 1$, $|\vec{c}| = |\vec{a}|$

Q. 5. If $\vec{a}$, $\vec{b}$ are two vectors such that $|\vec{a}+\vec{b}| = |\vec{a}|$, then prove that vectors $2\vec{a}+\vec{b}$ is perpendicular to vectors $\vec{b}$.

Q. 6. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors, such that $\vec{a}+\vec{b}+\vec{c} = \vec{0}$, and $|\vec{a}| = 3$, $|\vec{b}| = 5$, $|\vec{c}| = 7$, then prove that angle between $\vec{a}$ and $\vec{b}$ is 60°.

Q. 7. If $\vec{a} = i + j + k$ and $\vec{b} = j - k$, find a vector $\vec{c}$ such that $\vec{a}\times\vec{c} = \vec{b}$ and $\vec{a}\cdot\vec{c} = 3$
$Ans: \frac{1}{3}(5i + 2j + 2k)$

Q. 8. If for non zero vector $\vec{a}$, $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$ and $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$. Show that $\vec{b} = \vec{c}$

Q. 9. For any two vectors $\vec{a}$ & $\vec{b}$ prove that $|\vec{a}\times\vec{b}|^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} \end{vmatrix}$

Q. 10. Prove that the points A, B & C with position vectors $\vec{a}, \vec{b}$ & $\vec{c}$ respectively are collinear if and only if $\vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{a}\times\vec{b} = \vec{0}$

Q. 11. If three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ satisfies $\vec{a}+\vec{b}+\vec{c} = \vec{0}$, prove that $\vec{b}\times\vec{c} = \vec{c}\times\vec{a} = \vec{a}\times\vec{b}$

Q. 12. Using vectors prove that $(a_1b_1 + a_2b_2 + a_3b_3)^2 \le ({a_1}^2 + {a_2}^2 + {a_3}^2)({b_1}^2 + {b_2}^2 + {b_3}^2)$

Q. 13. If, $\vec{a}\times\vec{b} = \vec{c}\times\vec{d}$ and $\vec{a}\times\vec{c} = \vec{b}\times\vec{d}$. Show that $\vec{a} - \vec{d}$ is parallel to $\vec{b} - \vec{c}$

Q. 14. Find a unit vector in the plane of the vectors $i + 2j$ and $j + 2k$ which is perpendicular to the vector $2i + j + 2k$. $Ans: \frac{1}{5\sqrt{5}}(-5i - 6j + 8k)$

Q. 15. If A, B, C, D are four points in space, prove that
$|\overrightarrow{AB}\times\overrightarrow{CD} + \overrightarrow{BC}\times\overrightarrow{AD} + \overrightarrow{CA}\times\overrightarrow{BD}| = 4(area\ \Delta ABC)$

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12. Three – Dimensions

Q. 1. Find the point of intersection of the lines: $\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$; $\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$, if the lines are intersecting. *Ans*: $(1, 3, 2)$

Q. 2. Find the image of the point $(0, 2, 3)$ in the line $\frac{x+3}{5} = \frac{y-1}{2} = \frac{z+4}{3}$ *Ans*: $(4, 4, -5)$

Q. 3. Find the foot and length of perpendicular drawn from the point $(1, 2, 3)$ on the line $\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}$ *Ans*: $(3, 5, 9), 7\ units$

Q. 4. If a variable line in two adjacent positions has direction cosines l, m, n & $l + \delta l, m + \delta m, n + \delta n$, Show that the small angle $\delta\theta$ between the two positions is given by $\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2$.

Q. 5. Find the equations of the two straight lines through origin such that each line is intersecting the line $\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1}$ at an angle of $\frac{\pi}{3}$. *Ans*: $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$; $\frac{x}{-1} = \frac{y}{1} = \frac{z}{-2}$

Q. 6. A line with direction cosines proportional to $2, 7, -5$ is drawn to intersect the lines $\frac{x-5}{3} = \frac{y-7}{-1} = \frac{z+2}{1}$; $\frac{x+8}{-3} = \frac{y-3}{2} = \frac{z-6}{4}$. Find the coordinates of the point of intersection and the length intercepted on it. *Ans*: $(2, 8, -3), (0, 1, 2), \sqrt{78}$

Q. 7. Find the length and equation of shortest distance $(S.D)$ between the lines, $\vec{r} = (2\lambda + 3)i - (7\lambda + 15)j + (5\lambda + 9)k$ & $\vec{r} = (2\mu - 1)i + (1 + \mu)j + (9 - 3\mu)k$.
Ans: $4\sqrt{3}$; $\vec{r} = -i - j - k + 4t(i + j + k)$

Q. 8. Find the vector & cartesian equation of line through the point $(1, 2, 3)$ and perpendicular to the line $\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}$ *Ans*: $\vec{r} = i + 2j + 3k + \mu(2i + 3j + 6k)$; $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{6}$

Q. 9. Find the direction cosines of the line which are connected by the relation $l - 5m + 3n = 0$ and $7l^2 + 5m^2 - 3n^2 = 0$. *Ans*: $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$; $-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}$

Q. 10. Show that the lines $\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-1}{5}$; $\frac{x+2}{4} = \frac{y-1}{3} = \frac{z+1}{-2}$ do not intersect.

Q. 11. Show that the lines $\frac{x-3}{2} = \frac{y-15}{-7} = \frac{z-9}{5}$; $\frac{x+1}{2} = \frac{y-1}{1} = \frac{z-9}{-3}$ are skew lines.

Q. 12. Find the angle between the lines whose direction cosines are given by the relations $2l - m + 2n = 0$ & $mn + nl + lm = 0$ *Ans*: $\frac{\pi}{2}$

Q. 13. Find the equation of line which intersect the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$; $\frac{x+2}{1} = \frac{y-3}{2} = \frac{z+1}{4}$ and passes through the point $(1, 1, 1)$. *Ans*: $\frac{x-1}{3} = \frac{y-1}{10} = \frac{z-1}{17}$

Q. 14. Verify that $\frac{l_1+l_2+l_3}{\sqrt{3}}, \frac{m_1+m_2+m_3}{\sqrt{3}}, \frac{n_1+n_2+n_3}{\sqrt{3}}$ can be taken as the $d.c$'s of a line equally inclined to three mutually perpendicular lines with $d.c$'s : l_1, m_1, n_1 ; l_2, m_2, n_2; l_3, m_3, n_3

Q. 15. If the straight lines having $d.c$'s given by $al + bm + cn = 0$ and $fmn + gnl + hlm = 0$ are perpendicular, then show that , $\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$

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13. Linear Programming

Q. 1. A small manufacturer has employed 5 skilled men and 10 semi – skilled men and makes an article in two qualities, a deluxe model and an ordinary model. The making of a deluxe model requires 2 *hrs* of work by a skilled man and 2 *hrs* of work by a semi – skilled man. The ordinary model requires 1 *hr* by a skilled man and 3 *hrs* by a semi – skilled man. By union rule, no man may work for more than 8 *hrs* per day. The manufacturer gains *Rs* 15 on deluxe model and *Rs* 10 on ordinary model. How many of each type should be made in order to maximize his total daily profit?
Ans: *Max Profit* = *Rs*. 350, Deluxe model 10 *units* and Ordinary model 20 *units*

Q. 2. A toy company manufactures two types of dolls, A and B. Each doll of type B takes twice as long as to produce as one of type A. If the company produces only type A, it can make a maximum of 2000 dolls per day. The supply of plastic is sufficient to produce 1500 dolls per day. Type B requires a fancy dress which cannot be available for more than 600 per day. If the company makes profits of Rs 3 and Rs 5 per doll respectively on dolls A and B, how many of each should be produced per day in order to maximize the profit? $Ans:$ $Max\ Profit = Rs\ 5500,\ doll\ A = 1000, doll\ B = 500.$

Q. 3. A man owns a field of area $1000\ m^2$. He wants to plant trees in it. He has a sum of Rs 1400 to purchase young trees. He has the choice of two types of trees. Type A requires $10\ m^2$ of ground per tree and costs $Rs\ 20/tree$ and type B requires $20\ m^2$ of ground per tree and costs $Rs\ 25/tree$. When fully grown, type A produces an average of $20\ kg$ of fruits which can be sold at a profit of $Rs\ 2/kg$ and type B produces an average of $40\ kg$ of fruits which can be sold at a profit of $Rs\ 1.50/kg$. How many trees of each type should be planted to achieve maximum profit when the trees are fully grown? What is the maximum profit? $Ans: Max\ profit = Rs\ 3200, A = 20, B = 40.$

Q. 4. A young man rides his motorcycle at $25\ km/hr$, he has to spend $Rs\ 2/km$ on petrol, if he rides it at a faster speed of $40\ km/hr$, the petrol cost increases to $Rs\ 5/km$. He has Rs 100 to spend on petrol and wishes to find the maximum distance he can travel within $1\ hour$. Express this as an $L.P.P.$ and solve it.

$Ans: \frac{50}{3}\ km\ at\ 25\ km/hr\ \&\ \frac{40}{3}\ km\ at\ 40\ km/hr.$

Q. 5. Two tailors A & B, earn Rs 150 & Rs 200 per day, respectively. A can stitch 6 shirts and 4 pants, while B can stitch 10 shirts and 4 pants per day. Form the linear programming problem to minimize the labour cost to produce at least 60 shirts and 32 pants and solve it.

$Ans: Min\ Cost = Rs\ 1350, A\ for\ 5\ days\ and\ B\ for\ 3\ days.$

Q. 6. A house wife wishes to mix together two kinds of food I & II, in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of one kg of food is given below:

$Food$	$Vitamin\ A$	$Vitamin\ B$	$Vitamin\ C$
I	1	2	3
II	2	2	1

One kg of food I costs Rs 6 and one kg of food II costs Rs 10. Formulate and solve the above problem to find the least cost of the mixture which will produce the diet.

$Ans: Min\ Cost = Rs. 52, Food\ I = 2\ kgs, Food\ II = 4\ kgs.$

Q. 7. Vikas has been given two lists of problems from his mathematics teacher with the instructions to submit not more than 100 of them correctly solved for marks. The problems in the first list are worth 10 marks each and those in the second list are worth 5 marks each. Vikas knows from past experience that he requires on an average of 4 minutes to solve a problem of 10 marks and 2 minutes to solve a problem of 5 marks. He has other subjects to worry about; he cannot devote more than 4 hours to his mathematics assignment. With reference to manage his time in best possible way how many problems from each list shall he do to maximize his marks? $Ans: line\ segment\ joining\ (20,80), (60,0)$

Q. 8. A manufacturer of electronic circuits has a stock of 200 resistors, 120 transistors and 150 capacitors and is required to produce two types of circuits A and B. Type A requires 20 resistors, 10 transistors and 10 capacitors. Type B requires 10 resistors, 20 transistors and 30 capacitors. If the profit on type A circuit is Rs 50 and that on type B circuit is Rs 60. Formulate this problem as $L.P.P.$ and solve it graphically so that the manufacturer can maximize his profit. $Ans: Rs. \left(\frac{1640}{3}\right), A = \frac{28}{3}, B = \frac{4}{3}$

Q. 9. A manufacturer makes two types of toys A and B. Three machine are needed for this purpose and the time (in minutes) required for each toy on the machines is given below:

Toys	Machine I	Machine II	Machine III
A	20	10	10
B	10	20	30

The machines I, II and III are available for a maximum of $3\ hrs, 2\ hrs$ and $2\ hrs\ 30\ minutes$ respectively. The profit on each toy of type A is $Rs. 50$ and that of type B is $Rs. 60$. Formulate the above problem as $L.P.P$ and solve it graphically to maximize profit.

$Ans: Max\ Profit = Rs. 520, A = 8, B = 2$

Q. 10. A company produces two different products. One of them needs $\frac{1}{4}$ of an hour of assembly work per unit, $\frac{1}{8}$ of an hour in quality control work and $Rs\ 1.2$ in raw materials. The other product requires $\frac{1}{3}$ of an hour of assembly work per unit, $\frac{1}{3}$ of an hour in quality control work and $Rs\ 0.9$ in raw materials. Given the current availability of staff in the company, each day there is at most a total of 90 hours available for assembly and 80 hours for quality control. The first product described has a market value (sale price) of $Rs\ 9$ per unit and the second product described has a market value (sale price) of $Rs\ 8$ per unit. In addition, the maximum amount of daily sales for the first product is estimated to be 200 units. Formulate and solve graphically the LPP and find the maximum profit.

$Ans: Max.\ Profit = 2410, Product\ I = 200, Product\ II = 120$

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14. Probability

Q. 1. If A and B are two independent events such that $P(\bar{A} \cap B) = \frac{2}{15}$ and $P(\bar{B} \cap A) = \frac{1}{6}$, then find $P(A)$ and $P(B)$.

$Ans: P(A) = \frac{5}{6}, P(B) = \frac{4}{5}$ & $P(A) = \frac{1}{5}, P(B) = \frac{1}{6}$

Q. 2. Let A and B are two independent events. The probability of their simultaneous occurrence is $\frac{1}{8}$ and the probability that neither occur is $\frac{3}{8}$. Find $P(A)$ and $P(B)$.

$Ans: \frac{1}{4}, \frac{1}{2}$ & $\frac{1}{2}, \frac{1}{4}$

Q. 3. Probability of solving specific problem independently by A, B & C are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{6}$ respectively. If both try to solve the problem independently, find the probability that (i) the problem is solved (ii) exactly one of them solves the problem (iii) exactly two of them solves the problem

$Ans: (i)\frac{13}{18} (ii)\frac{17}{36} (iii)\frac{2}{9}$

Q. 4. Three defective bulbs are accidently mixed with seven good ones**.** If three bulbs are drawn**,** what is the average number of defective bulbs drawn**?** $Ans: 0.9$

Q. 5. Two cards are drawn without replacement from a well-shuffled deck of 52 cards**.** Determine the probability distribution of the number of face cards.

$Ans:$

X	0	1	2
$P(X)$	$130/221$	$80/221$	$11/221$

Q. 6. Two cards are drawn with replacement**,** from a well–shuffled pack of 52 cards**.** Find the probability distribution for number of face cards. Also find mean, variance and standard deviation for the number of face cards drawn. $Ans:$

X	0	1	2
$P(X)$	$100/169$	$60/169$	$9/169$

$mean = \frac{6}{13}, variance\ \frac{60}{169}, standard\ deviation = \frac{2\sqrt{15}}{13}$

Q. 7. Out of a group of 8 highly qualified doctors in a hospital, 6 are very kind and cooperative with their patients and so are very popular, while the other two remain reserved. For a health camp, three doctors are selected at random. Find the probability distribution of the number of very popular doctors.

Ans:

X	1	2	3
$P(X)$	$3/28$	$15/28$	$10/28$

Q. 8. There are 10 people in a group. Out of them 8 people are non –vegetarian, 3 people are selected at random. Write the probability distribution of vegetarian people.

Ans:

X	0	1	2
$P(X)$	$7/15$	$7/15$	$1/15$

Q. 9. A die is biased so that the even number is two times as likely to occur as odd number**.** If the die is tossed twice, find the probability distribution for the number of perfect squares.

Ans :

X	0	1	2
$P(X)$	$4/9$	$4/9$	$1/9$

Q. 10. Bag I contains 3 red and 4 black balls and Bag II contains 4 red and 5 black balls. Two balls are transferred from Bag I to Bag II and then a ball is drawn from Bag II. The ball so drawn is found to be red in colour. Find the probability that the transferred balls are black. $Ans: \frac{4}{17}$

Q. 11. In a test**,** an examinee either guesses or copies or knows the answer to a multiple-choice question with four choices**.** The probability that he makes a guess is $\frac{1}{3}$ and the probability that he copies the answer is $\frac{1}{6}$. The probability that his answer is correct**,** given that he copied it is $\frac{1}{8}$**.** Find the probability that he knew the answer to the question, given that he correctly answered it**.** $Ans: \frac{24}{29}$

Q. 12. A bag contains four balls. Two balls are drawn at random, and are found to be white. What is the probability that all balls are white? $Ans: \frac{3}{5}$

Q. 13. A man is known to speak truth 4 out of 5 times. He throws a pair of dice and reports it is a doublet. What is the probability that (i) actually it is a doublet (ii) He is speaking a lie? $Ans: (i)\frac{4}{9}\,(ii)\frac{5}{9}$

Q. 14. A letter is known to have come from $LONDON$ or $CLIFTON$. On the envelope just two consecutive letters ON are visible. What is the probability that the letter has come from $CLIFTON$.

$Ans: \frac{5}{17}$

Q. 15. A and B take turn to through a pair a die, the first to through 9 being awarded the prize. If A is first to throw the die, show that their chance of winning are in the ratio $9:8$.

Q. 16. If A and B are two independent events, then prove that the probability of occurrence of at least one of A and B is given by $1 - P(\bar{A}\,)P(\bar{B})$

Q. 17. A bag contains $(2n + 1)$ coins. It is known that n of these coins have a head on both its sides whereas the rest of the coins are fair. A coin is picked up at random from the bag and is tossed. If the probability that the toss results in a head is $\frac{31}{42}$ find the value of n. $Ans: n = 10$

Q. 18. A committee of 4 students is to be selected from a group consisting of 7 boys and 4 girls. Find the probability that there are exactly 2 boys in the committee given that at least one girl must be there in the committee. $Ans: \frac{126}{295}$

Q. 19. A and B throw a pair of dice alternately, A wins the game if he gets a total of 7 and B wins the game if he gets a total of 10. If A starts the game, then find the probability that B wins. $Ans: \frac{5}{17}$

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